ADMISSIBLE COORDINATES OF LICHNEROWICZ FOR THE SCHWARZSCHILD METRIC

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Abstract

Global harmonic coordinates for the complete Schwarzschild metric are found, for a more general case than a previous work by Quan [4]. The supplementary constant appearing, in addition to the mass, is related to the stress quadrupole moment by means of a singular energy–momentum tensor. Similar calculations are also carried out in q–harmonic coordinates.

1. - Introduction

Since the beginning of the General Relativity theory it is broadly used to handle harmonic coordinates for several purposes. In many cases they are used because of mathematical comfort reasons [1], and some other times used with the purpose of attributing them an special meaning [2], in particular they are considered to be a natural generalization of euclidean cartesian coordinates. Nevertheless, recently before no global system of harmonic and asymptotically cartesian coordinates had been showed for a kind of stelar model, that is to say, a system being well defined everywhere (interior and exterior) and with a C^1 metric on the surface of the star (admissible coordinates in the sense of Lichnerowicz [3]). One example of this have been provided by Quan Hui Liu [4]: the complete metric of Schwarzschild (interior and exterior) with $\mu_0 = 8/9$ (being μ_0 the dimensionless quotient between two times the mass of the star and its radius). It should be noticed that this special value of the parameter μ_0 leads to a simplification of the calculations but awkfully it also provides a model of star with divergencies in the pressure at the center of the star. This project forces to introduce two new constants (in addition to the mass) into the metric, one of them $Q_{\rm ext}$ at the exterior and another $Q_{\rm int}$ at the interior. These constants, which are defined with length dimension in the paper mentioned above, result to be proportional to the radius of the star and they are exempt from meaning by the author.

The aim of this article is double: In one hand, we are going to develop the Quan Hui Lin's programme but in a more realistic and general case for the parameter μ_0 ($\mu_0 < 8/9$). The resolution of the problem for an arbitrary value of that parameter leads

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to a not well known Heun's differential equation [5] instead of the simple hypergeometric equation appearing in the limiting case $\mu_0 = 8/9$. In the other hand, it is showed by use of the linear aproximation of vacuum and spherically symmetric Einstein's equation that the exterior constant $Q_{\rm ext}$ is closely related with the stress quadrupole moment of the source.

In Section 2 the complete Schwarzschild's metric is written in standard coordinates just to set the notation will be used, and the Darmois's matching conditions [6] are briefly reviewed by expliciting the first and the second fundamental forms on the surface of the star.

In Section 3 the change of coordinates is found, from standard ones to a general asymptotically cartesian harmonic coordinates, being fulfilled two conditions: the new sytem of coordinates should preserve the spherical symmetry and diagonal structure of the metric (written down in associated polar coordinates) and in adittion, the function relating both sets of coordinates must be C^1 on the surface of the star. Those conditions allow us to express the exterior and interior Quan's constants in terms of the change functions evaluated on the boundary. Then, it is proved that the new coordinates are admissible coordinates in the sense of Lichnerowicz [3], i.e. metric is C^1 on the surface.

Section 4 is devoted to interpret the exterior constant $Q_{\rm ext}$. In order to do that, it is firstly written down the multipole expansion of the exterior Schwarzschild metric in the new set of harmonic coordinates previously found (using the inverse of the radial coordinate, up to order five, as the parameter of the series). That expansion was already shown although in a slightly different way by one of us [7]. Secondly, it is made explicit the more general solution of the vacuum Einstein's equations with spherical symmetry in the linear approximation, and so, suitable comparison with the previous expansion can be established, which provides arguments to understand the significance of the new constants appearing in the metric. Finally, it is verified that this solution also proceeds from some singular (and pointlike defined) energy momentum tensor whose stress quadrupole moment results to be "proportional" to the exterior constant of Quan (which means that Quan's c! onstant is the factor multipling the tensorial expression of that multipole moment).

In Section 5 it is briefly showed the results obtained for the same project by using the so called q-harmonic coordinates, a variety of harmonic coordinates introduced by L. Bel [8]. The biggest part of the contents included here have already been obtained by J.M. Aguirregabiría [9] for the exterior case and by P. Teyssandier [10] for both exterior and interior cases. Those results, as well as some new other ones, are introduced by sort of comparison with the harmonic scenario and because they are until now not published completely.

The whole of the paper is completed with one Appendix which is devoted to explain the Heun's differential equation and it has been included because of the rather odd and at the same time fundamental equation involved. We think that it will make easy to the reader the understanding of Section 3.

2. - Complete model of Schwarzschild metric

The interior Schwarzschild metric has the following expression in standard polar coordinates $\{t\,,r\,,\theta\,,\varphi\}$

$$ds_I^2 = -\left[\frac{3}{2}\,\gamma^{1/2}(r_0) - \frac{1}{2}\,\gamma^{1/2}(r)\right]^2 dt^2 + \gamma^{-1}(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \qquad (1)$$

where the radial coordinate r is supposed to be restricted to the interval $[0, r_0]$, being r_0 the radius of the star. The following notation is used:

$$\gamma(r) \equiv 1 - \frac{r^2}{L^2} \quad , \quad \frac{1}{L^2} \equiv \frac{1}{3}\chi\rho \quad (r_0 < L) \quad , \quad \chi \equiv 8\pi \qquad (2)$$

being ρ the energy density (constant) of the star and where we deal with geometrized units ($G = c = 1$). As it is known, the pressure of the model is given as a function of the radius by the following expression

$$p(r) = \rho\frac{\gamma^{\frac{1}{2}}(r) - \gamma^{\frac{1}{2}}(r_0)}{3\gamma^{\frac{1}{2}}(r_0) - \gamma^{\frac{1}{2}}(r)} \qquad (3)$$

which provides the following values for the pressure at the surface and the center of the star respectively

$$\begin{cases} p(r_0) = 0 \\ \\ p(0) \equiv p_c = \rho\dfrac{1 - \gamma^{\frac{1}{2}}(r_0)}{3\gamma^{\frac{1}{2}}(r_0) - 1} \end{cases} \qquad (4)$$

Since the value p_c of the pressure at the center of the star must be finite, it relates the radius of the star r_0 and the density parameter L by means of the following restriction

$$3\gamma^{\frac{1}{2}}(r_0) - 1 > 0 \quad \Rightarrow \quad \frac{r_0}{L} < \frac{2\sqrt{2}}{3} \qquad (5)$$

The exterior Schwarzschild metric is written down in the same set of standard polar coordinates as follows ($r \geq r_0$)

$$ds_E^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \qquad (6)$$

where m represents the mass of the star, and it is related with the density ρ and the radius r_0 by the known expression

$$m \equiv \frac{4}{3}\pi r_0^3\,\rho \qquad \left(\Leftrightarrow \;\; \frac{2m}{r_0} = \frac{r_0^2}{L^2} < \frac{8}{9}\right) \tag{7}$$

The expressions showed above for the Schwarzschild metric (interior and exterior) are so that the Darmois' matching conditions [6] are automaticaly fulfilled on the surface of the star $\Sigma: \; r = r_0$. These conditions, that is to say, the continuity of both the first and the second fundamental forms will be symbolically denoted as follows

$$\begin{cases} \mathrm{I}: \left[h_{ab}\right]_\Sigma = 0 \\ \mathrm{II}: \left[K_{ab}\right]_\Sigma = 0 \end{cases} \quad , \quad (a, b, \ldots = 0, 2, 3 = t, \theta, \varphi) \tag{8}$$

In what follows we make explicit the expression of the fundamental forms to show the requirements provided by (8).

- *The first fundamental form*

$$h_{ab}(p) \equiv g_{\alpha\beta}[x(p)]\, e^\alpha_{a/}\, e^\beta_{b/} \tag{9}$$

being $e^\alpha_{a/}$ the tangent vectors to the surface Σ

$$e^\alpha_{a/}(p^b) \equiv \frac{\partial x^\alpha}{\partial p^b} \qquad \left(\Sigma: \; x^0 = p^0 = t \;,\; x^2 = p^2 = \theta \;,\; x^3 = p^3 = \varphi\right) \tag{10a}$$

which leads to

$$(h_{ab}) = \begin{pmatrix} g_{tt} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{\varphi\varphi} \end{pmatrix} \tag{10b}$$

i.e., g_{tt} must be continuous, being g_{rr} without restriction (let us note that coordinates verify the euclidean sphere condition everywhere: $g_{\varphi\varphi} = g_{\theta\theta}\sin^2\theta = r^2\sin^2\theta$). We will see below that the continuity of g_{rr} is imposed by the continuity of the second fundamental form.

- *The second fundamental form*

$$K_{ab} \equiv -l_\mu e^\rho_{a/} \nabla_\rho e^\mu_{b/} = -l_\alpha \left(\frac{\partial e^\alpha_{a/}}{\partial p^b} + \Gamma^\alpha_{\lambda\mu} e^\lambda_{a/} e^\mu_{b/}\right) = -\Gamma^r_{ab} \tag{11}$$

where the notation $l_\alpha \equiv \partial_\alpha(r - r_0)$ has been used to denote the normal vector to the surface Σ

$$(K_{ab}) = \frac{1}{2} g^{rr} \begin{pmatrix} \partial_r g_{tt} & 0 & 0 \\ 0 & \partial_r g_{\theta\theta} & 0 \\ 0 & 0 & \partial_r g_{\varphi\varphi} \end{pmatrix} \tag{12}$$

It is deduced from (12) that both g_{rr} and $\partial_r g_{tt}$ must be continuous. As a summary, only the discontinuity of $\partial_r g_{rr}$ is allowed which following (1) and (6) turns out to be

$$[\partial_r g_{rr}]_\Sigma \equiv \partial_r g^E_{rr}(r_0) - \partial_r g^I_{rr}(r_0) = -3 \frac{r_0}{L^2} \gamma^{-2}(r_0) \tag{13}$$

It is then obvious that the set of standard polar coordinates $\{t\,, r\,, \theta\,, \varphi\}$ describing the complete model of Schwarzschild are not admissible in the sense of Lichnerowicz, since the metric is not C^1.

3. - Global asymptotically cartesian harmonic coordinates

The aim of this section is to find a global system of harmonic coordinates for the complete model of Schwarzschild, which must be asymptotically cartesian and so that the components of the metric are C^1 on the boundary $\Sigma : \, r = r_0$. Let us consider the generic form of the Schwarzschild metric in standard coordinates (1),(6)

$$ds^2 = g_{tt}(r) dt^2 + g_{rr}(r) dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{14}$$

Now, we perform a change of coordinates

$$\{t, r, \theta, \varphi\} \equiv \{x^\alpha\} \longrightarrow \{\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}\} \equiv \{x^{\tilde{\alpha}}\} \tag{15}$$

verifying the following conditions

$$\begin{cases} \partial_\lambda \left(\sqrt{-g}\, g^{\lambda\mu} \partial_\mu x^{\tilde{\alpha}} \right) = 0 \\ \\ \lim\limits_{\tilde{r} \to \infty} g_{\tilde{\alpha}\tilde{\beta}} = \eta_{\alpha\beta} \equiv \mathrm{diag}(-1, +1, +1, +1) \end{cases} \tag{16}$$

which represents harmonicity and cartesian behaviour respectively and where $\tilde{r} \equiv \sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2}$.

The easiest way to solve the problem is by using polar coordinates $\{t, \tilde{r}, \tilde{\theta}, \tilde{\varphi}\}$ associated to the harmonic ones $\{x^{\tilde{\alpha}}\}$

$$\begin{cases} \tilde{x} + i\, \tilde{y} = \tilde{r} \sin \tilde{\theta} \, e^{i \tilde{\varphi}} \\ \\ \tilde{z} = \tilde{r} \cos \tilde{\theta} \end{cases} \tag{17}$$

It is found that, in order to preserve the diagonal structure as well as the symmetries mentioned, the change of coordinates (15) should be as follows

$$\tilde{t} = t \quad , \quad \tilde{r} = f(r) \quad , \quad \tilde{\theta} = \theta \quad , \quad \tilde{\varphi} = \varphi$$
$$\left(\lim_{r \to \infty} \frac{f(r)}{r} = 1 \right) \tag{18}$$

Hencefore, the equation (16) turns out to be the following second order linear differential equation for the function $f(r)$:

$$\frac{d}{dr}\left(r^2 \sqrt{\frac{-g_{tt}}{g_{rr}}} \frac{df}{dr} \right) - 2\sqrt{-g_{tt}\, g_{rr}}\, f = 0 \tag{19}$$

Let us see now the form of this equation at the interior and the exterior of the star.

A) Exterior problem

For the given expressions of the exterior Schwarzschild metric (6), the equation (19) is written as follows

$$r\,(r - 2m) f''(r) + 2(r - m) f'(r) - 2f(r) = 0 \tag{20}$$

whose general solution is the following [4]

$$\tilde{r} = f_{\rm ext}(r) = Q_1 (r - m) + Q_{\rm ext}\, \tilde{g}(r) \tag{21}$$

where Q_1 y $Q_{\rm ext}$ are arbitrary constants and the function $\tilde{g}(r)$ is defined

$$\tilde{g}(r) \equiv (r - m) \log \left(1 - \frac{2m}{r} \right) + 2m \tag{22}$$

Looking at the asymptotically condition we have

$$f_{\rm ext}(r \to \infty) \sim Q_1 r \left(1 - \frac{m}{r} \right) + Q_{\rm ext} \frac{2m^2}{r} \sim Q_1 r \tag{23}$$

and then, the constant Q_1 must be equal to one ($Q_1 = 1$).

B) Interior problem

By introducing now the expression (1) of the interior metric into the generic equation (19) we obtain the following differential equation for the function $f(r)$

$$r^2 \gamma(r)\, f''(r) + r \left[2\gamma(r) - \frac{r^2}{L^2} \frac{3\gamma^{\frac{1}{2}}(r_0) - 2\gamma^{\frac{1}{2}}(r)}{3\gamma^{\frac{1}{2}}(r_0) - \gamma^{\frac{1}{2}}(r)} \right] f'(r) - 2f(r) = 0 \tag{24}$$

which is at first sight no easy to handle with. Nevertheless, the change of the independent variable

$$r \longrightarrow x = 1 - \gamma^{\frac{1}{2}}(r) \; : \; \begin{cases} r \in \big[0\,, r_0\big] \\ \\ x \in \big[0\,, x_0 \equiv 1 - \gamma^{\frac{1}{2}}(r_0)\big] \end{cases} \tag{25}$$

leads to the following equation

$$f''(x) + \frac{4x^2 - (5+3b)x + 3b}{(x-b)x(x-2)} f'(x) - \frac{2}{x^2(x-2)^2} f(x) = 0 \tag{26}$$

where primes denote now derivatives with respect to the new variable x and the following definition has been used

$$b \equiv 1 - 3\gamma^{\frac{1}{2}}(r_0) \tag{27}$$

Let us notice at this moment what restrictions implies the upper limit of the parameter μ_0:

$$0 < \mu_0 < \frac{8}{9} \quad \Rightarrow \quad \begin{cases} 0 < x_0 < \dfrac{2}{3} \\ \\ -2 < b < 0 \end{cases} \tag{28}$$

This new differential equation (26) have four singular points which are all of them regular

$$\{x = b\,, x = 0\,, x = 2\,, x = \infty\} \tag{29}$$

Now, since we are interested on those solutions with a well behaviour into the interval $[0, x_0]$, we proceed to look for analytical solutions at the origin as a kind of Frobenius' series. This procedure leads to the uniqueness of the analytical series with exponent $E_f = 1/2$, but it should be constructed by means of a recursive expression highly unpleasent. For this reason we have choosen to reduce the equation (26) into the Heun's canonical form [5] (see Appendix) which allows to obtain the required solution by using some properties of that equation. The final result for the general and analytical solution at the origin is the following convergent expression into the interval $[0, x_0]$

$$f_{\rm int}(r) = Q_{\rm int}\, r\, H\left(\frac{2}{b}, 3 + \frac{1}{b}; 4, 1, \frac{5}{2}, 1; \frac{x}{b}\right) \tag{30}$$

being $Q_{\rm int}$ an arbitrary constant and where $H(a, q; \alpha, \beta, \gamma, \delta; z)$ is the called Heun's series defined as follows

$$H(a, q; \alpha, \beta, \gamma, \delta; z) = 1 + \sum_{n=1}^{\infty} c_n\, z^n \tag{31}$$

with the recursive law $(n = 0, 1, 2, \ldots)$

$$
\begin{aligned}
& a(n+2)(\gamma+n+1)c_{n+2} \\
& -(n+1)\left[\alpha+\beta-\delta+n+1+a(\gamma+\delta+n)+\frac{q}{n+1}\right]c_{n+1} \\
& +\left[n(\alpha+\beta+n)+\alpha\beta\right]c_n=0
\end{aligned}
$$

$$
c_0=1 \quad , \quad c_1=\frac{q}{a\gamma} \tag{32}
$$

which is convergent iff

$$
|a|\geq 1 \quad , \quad |z|<1 \tag{33}
$$

Let us notice that in our case we have

$$
1<|\frac{2}{b}|<\infty \quad , \quad |\frac{x}{b}|<1 \quad \text{si} \quad \mu_0<\frac{3}{4} \tag{34}
$$

i.e., the series is convergent everywhere into the interior of the star only if its radius is larger than the Schwarzschild radius by a factor 4/3. For example, a typical neutron star of 1.4 solar masses and a radius of $2\times 10^4\, m$ gives a parameter

$$
\mu_0=\frac{2m}{r_0}=\frac{1.4\times 1.5}{20}=0.15<<0.75 \tag{35}
$$

So, $\mu_0<3/4$ is a large upper limit which does not imply restrictions to a realistic star model, and hence, the solution (30) becomes widely general.

By writting the solution (30) as a power series in the parameter r/L we find the following expression

$$
\begin{aligned}
f_{\text{int}}(r)=& Q_{\text{int}}\, r\left(1+\frac{1+3b}{10\,b}\,\frac{r^2}{L^2}+\frac{8+19b+45b^2}{280\,b^2}\,\frac{r^4}{L^4}\right. \\
& \left.+\frac{152+424b+729b^2+1575b^3}{15120\,b^3}\,\frac{r^6}{L^6}+\cdots\right)
\end{aligned} \tag{36}
$$

- *Quan's case*

$$
\mu_0\equiv\frac{r_0^2}{L^2}=\frac{2m}{r_0}=\frac{8}{9} \qquad (\Leftrightarrow b=0) \tag{37}
$$

For this case the solution (30),(36) is strongly divergent and we have to come back into the original differential equation (24) taking the parameter b to be zero ($3\gamma^{\frac{1}{2}}(r_0)=1$). That resulting equation only has three regular and singular points (one of them at the infinity) and hence, it reduces to an hypergeometric equation by using the standard procedure (see [4] for final result).

C) Admissible coordinates of Lichnerowicz

We are going to prove that it is possible to set the constants $Q_{\rm int}$ and $Q_{\rm ext}$ in such a way that the components of the metric and its derivatives, written in the new set of coordinates, are continuous on the surface of the star ($r = r_0$). In order to do that we only require the function $f(r)$ defining the change of coordinates to be C^1 in that boundary, i.e.,

$$[f]_\Sigma = [f']_\Sigma = 0 \quad \Leftrightarrow \quad \begin{cases} f_{\rm ext}(r_0) = f_{\rm int}(r_0) \\ \\ f'_{\rm ext}(r_0) = f'_{\rm int}(r_0) \end{cases} \tag{38}$$

which in agreement with (21-22), (30) can be translated into the following relations

$$\begin{aligned} r_0 - m + \tilde{g}(r_0)\, Q_{\rm ext} &= r_0 \tilde{h}(r_0)\, Q_{\rm int} \\ 1 + \tilde{g}'(r_0)\, Q_{\rm ext} &= \left[\tilde{h}(r_0) + r_0 \tilde{h}'(r_0)\right] Q_{\rm int} \end{aligned} \tag{39}$$

being

$$\tilde{h}(r) \equiv H\left(\frac{2}{b}, 3 + \frac{1}{b}; 4, 1, \frac{5}{2}, 1; \frac{x}{b}\right) \tag{40}$$

The set of equations (39) constitutes an algebraic linear system for the unknown variable $Q_{\rm int}$ and $Q_{\rm ext}$,whose solution is

$$\begin{cases} Q_{\rm ext} = \dfrac{(r_0 - m)[\tilde{h}(r_0) + r_0 \tilde{h}'(r_0)] - r_0 \tilde{h}(r_0)}{r_0 \tilde{h}(r_0)\tilde{g}'(r_0) - [\tilde{h}(r_0) + r_0 \tilde{h}'(r_0)]\tilde{g}(r_0)} \\ \\ Q_{\rm int} = \dfrac{(r_0 - m)\tilde{g}'(r_0) - \tilde{g}(r_0)}{r_0 \tilde{h}(r_0)\tilde{g}'(r_0) - [\tilde{h}(r_0) + r_0 \tilde{h}'(r_0)]\tilde{g}(r_0)} \end{cases} \tag{41}$$

which is well defined since the denominator of both expressions is trivially different from zero. We give now the firsts terms of the expansion in power series of the parameter μ_0 for the values of those constants

$$\begin{cases} 2mQ_{\rm ext} = r_0\left(\dfrac{12}{35} - \dfrac{4}{21}\mu_0 - \dfrac{58}{1155}\mu_0^2 - \dfrac{136}{5005}\mu_0^3 + \cdots\right) \\ \\ Q_{\rm int} = 1 - \dfrac{3}{4}\mu_0 - \dfrac{1}{16}\mu_0^2 - \dfrac{1}{96}\mu_0^3 + \dfrac{1439}{134400}\mu_0^4 \cdots \end{cases} \tag{42}$$

It is worthwhile to notice that these values have nothing to do with those obtained by Quan for the case $\mu_0 = 8/9$, since our results require an upper limit for the parameter $\mu_0 < 3/4$, and so the comparison is over.

Let us check now that this choice of constants leads to a C^1 metric, written in harmonic polar coordinates, on the boundary $r = r_0$. By making use of the change (15-18) we have

$$ds^2 = g_{tt}(r)dt^2 + \frac{g_{rr}(r)}{f'^2(r)}d\tilde{r}^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad , \quad [r = r(\tilde{r})] \tag{43}$$

Obviously, all the components of the metric as well as the derivatives of g_{tt}, $g_{\theta\theta}$ and $g_{\varphi\varphi}$ are continuous. With regarding to the derivative of $g_{\tilde{r}\tilde{r}}$ it turns out to be

$$[\partial_{\tilde{r}} g_{\tilde{r}\tilde{r}}]_\Sigma = \frac{1}{f'} \left[\left(\frac{g_{rr}}{f'^2} \right)' \right]_\Sigma = \frac{1}{f'^4} \Big([\partial_r g_{rr}]_\Sigma \, f'(r_0) - 2 g_{rr}(r_0) \, [f'']_\Sigma \Big) \tag{44}$$

But, taking into account the differential equations (20) and (24) for the exterior and the interior respectively , we have that

$$[f'']_\Sigma = -\frac{3}{2} \frac{r_0}{L^2} \gamma^{-1}(r_0) \, f'(r_0) \tag{45}$$

and therefore, since the discontinuity of $\partial_r g_{rr}$ is given by (13), it is finaly obtained that

$$[\partial_{\tilde{r}} g_{\tilde{r}\tilde{r}}]_\Sigma = 0 \tag{46}$$

as we wanted to prove. Hence, the harmonic coordinates obtained with the constants (41-42) are admissible coordinates in the sense of Lichnerowicz.

4. - Interpretation of the Quan's exterior constant

A) Multipole expansion

As it was already mentioned at the Introduction the purpose of this Section is to look for the physical meaning of the constant $Q_{\rm ext}$ associated with the choice of global harmonic coordinates given in the previous Section. We first carry on a multipole expansion of the exterior Scwarzschild metric in that coordinates, i.e., a power series in the inverse of the radial coordinate $\tilde{r}$. This expansion can be taken, up to order five, from a previous paper [7], where it appears for other purposes. For this time we write the metric in a base slightly different usefull for the following calculations.

$$ds^2 = T(\tilde{r}) \, dt^2 + \Big[A(\tilde{r}) \, \delta_{ij} + B(\tilde{r}) \, n_i n_j \Big] d\tilde{x}^i d\tilde{x}^j \tag{47}$$

being

$$n^i \equiv \frac{\tilde{x}^i}{\tilde{r}} \qquad \big(n_i \equiv \delta_{ij} n^j \big) \tag{48}$$

$$T \equiv g_{tt}(r) \quad , \quad A \equiv \frac{r^2}{\tilde{r}^2} \quad , \quad B \equiv \frac{g_{rr}(r)}{f'^2(r)} - \frac{r^2}{\tilde{r}^2} \qquad \Big[r = r(\tilde{r}) \Big] \tag{49}$$

By using these definitions and the results from [7] the following expansion up to the order $1/\tilde{r}^5$ is obtained

$$T = -1 + 2\frac{m}{\tilde{r}} - 2\frac{m^2}{\tilde{r}^2} + 2\frac{m^3}{\tilde{r}^3} - 2\frac{Km + m^4}{\tilde{r}^4} + 2\frac{2Km^2 + m^5}{\tilde{r}^5} + \cdots$$

$$A = 1 + 2\frac{m}{\tilde{r}} + \frac{m^2}{\tilde{r}^2} + 2\frac{K}{\tilde{r}^3} + 2\frac{Km}{\tilde{r}^4} + \frac{6}{5}\frac{Km^2}{\tilde{r}^5} + \cdots \tag{50}$$

$$B = \frac{m^2}{\tilde{r}^2} + 2\frac{-3K + m^3}{\tilde{r}^3} + 2\,\frac{-6Km + m^4}{\tilde{r}^4} + 2\frac{-9Km^2 + m^5}{\tilde{r}^5} + \cdots$$

where a new constant K is introduced, and it is related with $Q_{\rm ext}$ as follows

$$K \equiv \frac{1}{3}m^2\, Q_{\rm ext} \tag{51}$$

B) Linear approximation in harmonic coordinates

As it is well known [11] the Einstein's equations can be written in the following way

$$\partial_{\lambda\mu}(\mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\lambda\mu} - \mathfrak{g}^{\alpha\lambda}\mathfrak{g}^{\beta\mu}) = 16\pi G(-g)\left(T^{\alpha\beta} + t_L^{\alpha\beta}\right) \tag{52}$$

being $\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta}$ the contravariant metric density, $g \equiv \det(g_{\alpha\beta})$ the determinant of the metric and $t_L^{\alpha\beta}$ energy-momemtum pseudeotensor of Landau–Lifshitz. One of the procedures used in the past to solve these equations is the so called perturbative postminkowskian algorithm [12]; starting from a system of asymptotically cartesian coordinates, this method consists of looking for a solution as a formal power series in the gravitational constant G:

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \sum_{n=1}^{\infty} G^n \overset{n}{\mathfrak{h}}{}^{\alpha\beta} \tag{53}$$

and taking for the coordinates the harmonicity condition

$$\partial_\alpha \mathfrak{g}^{\alpha\beta} = 0 \tag{54}$$

In particular this procedure has been of great interest for the study of some aspects of the gravitational radiation [13]. We are now dealing with a vacuum stationary scenario, that is, $\mathfrak{h}^{\alpha\beta}$ is independent of time, therefore the equations resulting in the linear approximation are the following

$$\begin{cases} \Delta\ \overset{1}{\mathfrak{h}}{}^{\lambda\mu} = 0 \\ \partial_i\ \overset{1}{\mathfrak{h}}{}^{i\mu} = 0 \end{cases} \tag{55}$$

whose general solution can be written in the following way (see for instance [13])

$$\overset{1}{\mathfrak{h}}{}^{\lambda\mu} = \overset{1}{\mathfrak{h}}{}^{\lambda\mu}_{\text{can}} + \partial^\lambda w^\mu + \partial^\mu w^\lambda - \eta^{\lambda\mu}\partial_\rho w^\rho \quad , \quad (\Delta\, w_\lambda = 0) \tag{56}$$

where the canonical part $\left(\overset{1}{\mathfrak{h}}{}^{\lambda\mu}_{\text{can}}\right)$ is given by

$$\begin{cases} \overset{1}{\mathfrak{h}}{}^{00}_{\text{can}} = -4\displaystyle\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\sum_{m=-l}^{+l} M_l^m\, Y_l^m(\theta,\varphi) \\ \overset{1}{\mathfrak{h}}{}^{0j}_{\text{can}} = +4\displaystyle\sum_{l=1}^{\infty}\frac{1}{r^{l+1}}\sum_{m=-l}^{+l} S_l^m\, Y_{l,l}^{m,j}(\theta,\varphi) \\ \overset{1}{\mathfrak{h}}{}^{jk}_{\text{can}} = 0 \end{cases} \tag{57}$$

being M_l^m and S_l^m the static and dynamical multipole moments of Geroch–Hansen [14] (up to a numerical constant factor), and where the usual notation was used for spherical harmonics. Regarding to the "gauge" part of the solution (terms involving the function w^ρ in (56)) we have the following expressions

$$w^0 = \sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\sum_{m=-l}^{+l} C_l^m Y_l^m(\theta,\varphi) \tag{58a}$$

$$\begin{aligned} w^k = {} & \sum_{l=1}^{\infty}\frac{1}{r^{l}}\sum_{m=-l}^{+l} H_l^{m,k}\, Y_l^m(\theta,\varphi) + \sum_{l=1}^{\infty}\frac{1}{r^{l+1}}\sum_{m=-l}^{+l} J_l^m\, Y_{l,l}^{m,k}(\theta,\varphi) \\ & - \sum_{l=0}^{\infty}\frac{l+1}{r^{l+2}}\sum_{m=-l}^{+l} K_l^m\, Y_l^m(\theta,\varphi)\, n^k + \sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\sum_{m=-l}^{+l} K_l^m\, \partial^k Y_l^m(\theta,\varphi) \end{aligned} \tag{58b}$$

where C_l^m, $H_l^{m,k}$, J_l^m y K_l^m are new constants (scalar or vectorial ones) whose meanings have not been discussed yet.

The general solution of the equations (55) with spherical symmetry is obtained from the previous expressions (57-58) by restricting ourself to the monopolar terms, i.e, by taking into account only $l = 0$ in the above series:

$$\begin{cases} \overset{1}{\mathfrak{h}}{}^{00}_{\text{can}} = -4\dfrac{m}{r} & \left(m \equiv M_0^0\right) \\ \overset{1}{\mathfrak{h}}{}^{0j}_{\text{can}} = 0 \\ \overset{1}{\mathfrak{h}}{}^{ij}_{\text{can}} = 0 \end{cases} \quad , \quad \begin{cases} w^0 = \dfrac{C}{r} & \left(C \equiv C_0^0\right) \\ w^j = -\dfrac{K}{r^2}n^j & \left(K \equiv K_0^0\right) \end{cases} \tag{59}$$

expressions which in agreement with (56) lead to the following solution

$$\begin{cases} \overset{1}{\mathfrak{h}}{}^{00} = -4\dfrac{m}{r} \\ \overset{1}{\mathfrak{h}}{}^{0j} = \partial^j w^0 \\ \overset{1}{\mathfrak{h}}{}^{ij} = -2\dfrac{K}{r^3}\delta^{ij} + 6\dfrac{K}{r^3}n^i n^j \end{cases} \tag{60}$$

Finally, the linear order of the metric is obtained by using of the known relation

$$\overset{1}{g}_{\lambda\mu} = -\overset{1}{\mathfrak{h}}_{\lambda\mu} + \frac{1}{2}\overset{1}{\mathfrak{h}}\,\eta_{\lambda\mu} \qquad \left(\overset{1}{\mathfrak{h}} \equiv \eta_{\lambda\mu}\,\overset{1}{\mathfrak{h}}{}^{\lambda\mu}\right) \tag{61}$$

which turns out to be explicitely

$$\begin{cases} \overset{1}{g}_{00} = 2\dfrac{m}{r} \\ \overset{1}{g}_{0j} = \partial_j w^0 = 0 \quad , \quad (C = 0) \\ \overset{1}{g}_{jk} = \left(2\dfrac{m}{r} + 2\dfrac{K}{r^3}\right)\delta_{jk} - 6\dfrac{K}{r^3}n_j n_k \end{cases} \tag{62}$$

where we have taken $C = 0$ since $\overset{1}{g}_{0j}$ is a gradient and so, it can be omitted (static condition).

Let us notice that, as it should be, the metric (62) contains all the linear terms of the Schwarzschild metric written in harmonic coordinates (47),(50) shown in the previous Section. This detail, which might seem to be insignificant, turns out to be the relevant point to understand the meaning of the constant K, as we will see now.

C) Spherically symmetric point-like source

A significant feature of the metric (62) is that it turns out to be a solution of the linearized Eisntein's equation with the following singular energy-momentum tensor (order zero) at the right hand side of that equation,

$$\begin{cases} \overset{0}{T}{}^{00} = m\,\delta(\vec{x}) \\ \\ \overset{0}{T}{}^{0j} = 0 \\ \\ \overset{0}{T}{}^{ij} = \dfrac{1}{2}K\left[\delta^{ij}\delta^{kl} - \delta^{i(k}\delta^{l)j}\right]\partial_{kl}\delta(\vec{x}) \end{cases} \tag{63}$$

So, the equations look like the following ones

$$\begin{cases} \Delta\,\overset{1}{\mathfrak{h}}{}^{00} = 16\pi m\,\delta(\vec{x}) \\ \\ \Delta\,\overset{1}{\mathfrak{h}}{}^{ij} = 8\pi K\left[\delta^{ij}\Delta\delta(\vec{x}) - \partial^{ij}\delta(\vec{x})\right] \end{cases} \tag{64}$$

and we can solve them by using the Poisson integrals to give

$$\begin{cases} \overset{1}{\mathfrak{h}}{}^{00}(\vec{x}) = -4m\displaystyle\int\frac{\delta(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y} = -4\frac{m}{r} \\ \\ \overset{1}{\mathfrak{h}}{}^{ij}(\vec{x}) = +2K\displaystyle\int\frac{\partial^{ij}\delta(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y} = -2\frac{K}{r^3}\delta^{ij} + 6\frac{K}{r^3}n^in^j \end{cases} \tag{65}$$

which is just the metric density obtained before (60).

Let us now calculate the *stress quadrupolar moment* of this engergy-momentum tensor (63)

$$\begin{aligned} \int x^px^q\,\overset{0}{T}{}^{ij}\,d^3\vec{x} &= \frac{1}{2}K\left[\delta^{ij}\delta^{kl} - \delta^{i(k}\delta^{l)j}\right]\int x^px^q\partial_{kl}\delta(\vec{x})\,d^3\vec{x} \\ &= K\left[\delta^{ij}\delta^{pq} - \delta^{i(p}\delta^{q)j}\right] \end{aligned} \tag{66}$$

and we see that the constant K and hencefore $Q_{\rm ext}$, is related with stress quadrupolar moment of the source. In our opinion, this is just a first argument to understand the role playing this kind of constants which are over from the gauge of Thorne when the standard multipole moments are analysed. We expect to develop in the future a coherent theory that can justify completely the interpretation done.

5. - Global asymptotically cartesian q-harmonic coordinates

In Section 3 we have looked for a global system of harmonic coordinates which are asymptotically cartesian and admissible in the sense on Lichnerowicz. This project can also be solved for a system of the so called q-harmonic coordinates. These coordinates were introduced by Bel [8] to analyze frames of reference in General Relativity as congruences of time-like curves. Although the following results are due to other athors, we will show them briefly because they are not completely published yet.

Only q-harmonic coordinates hold its whole sense associated with the also called q-harmonic congruences [8]. Since the Schwarzschild metric admits as a q-harmonic congruence the time Killing congruence, we restrict ourself to remind the definition of q-harmonic coordinates for this case. The change of coordinates from the standard ones is denoted by

$$\{r, \theta, \varphi\} \equiv \{x^i\} \longrightarrow \{\bar{x}, \bar{y}, \bar{z}\} \equiv \{x^{\bar{i}}\} \tag{67}$$

with the q-harmonicity condition

$$\begin{cases} \partial_i \left(\sqrt{\hat{g}}\, g^{ij} \partial_j x^{\bar{k}} \right) = 0 \\ \\ \lim\limits_{\bar{r} \to \infty} g_{\bar{i}\bar{j}} = \delta_{ij} \end{cases} \tag{68}$$

where $\bar{r} \equiv \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}$ and $\hat{g}$ represents the determinant of the spatial metric of the quotient space defined by the Killing congruence.

As well as in the harmonic case, associated polar coordinates are defined by

$$\begin{cases} \bar{x} + i\bar{y} = \bar{r} \sin\bar{\theta}\, e^{i\bar{\varphi}} \\ \\ \bar{z} = \tilde{r} \cos\bar{\theta} \end{cases} \tag{69}$$

$$\bar{r} = f(r) \quad , \quad \bar{\theta} = \theta \quad , \quad \bar{\varphi} = \varphi \tag{70}$$

$$\left(\lim_{r \to \infty} \frac{f(r)}{r} = 1 \right)$$

and it leads to the following differential equation

$$\frac{d}{dr} \left(\frac{r^2}{\sqrt{g_{rr}}} \frac{df}{dr} \right) - 2\sqrt{g_{rr}}\, f = 0 \tag{71}$$

A) Exterior problem (q–harmonic coordinates)

The previous differential equation (71) is written in this case as follows

$$r\,(r-2m)f''(r)+2(r-\frac{3}{2}m)f'(r)-2f(r)=0 \tag{72}$$

whose general solution was already obtained by Aguirregabiría [9]

$$\bar{r}=f_{\rm ext}(r)=J_1\left(r-\frac{3}{2}m\right)+J_2\,\bar{g}(r) \tag{73}$$

where J_1 and J_2 are constants and the function $\bar{g}(r)$ is defined by

$$\bar{g}(r)\equiv\left(r-\frac{1}{2}m\right)\sqrt{1-\frac{2m}{r}} \tag{74}$$

with the following asymptotic behaviour

$$\bar{g}(r)=r\left(1-\frac{3}{2}\frac{m}{r}-\frac{1}{4}\frac{m^3}{r^3}+\cdots\right) \tag{75}$$

and so, according with (73) we have

$$J_1+J_2=1 \tag{76}$$

B) Interior problem (q–harmonic coordinates)

For this case the differential equation (71) looks like

$$r^2\left(1-\frac{r^2}{L^2}\right)f''(r)+r\left(2-\frac{3r^2}{L^2}\right)f'(r)-2f(r)=0 \tag{77}$$

whose unique analytical solution at the origin was already obtained by Teyssandier [10]

$$\begin{aligned} f_{\rm int}(r) &= P\,\frac{r}{L}\,F\left(\frac{1}{2},\frac{3}{2},\frac{5}{2};\frac{r^2}{L^2}\right)\\ &= P\,\frac{r}{L}\left(1+\frac{3}{10}\frac{r^2}{L^2}+\frac{9}{56}\frac{r^4}{L^4}+\frac{5}{48}\frac{r^6}{L^6}+\cdots\right)\end{aligned} \tag{78}$$

being P an arbitrary constant and where $F(a,b,c,;x)$ represents the usual hypergeometric function whose first terms of its power expansion are showed for clarity of expression.

C) Continuity on the boundary Σ

As well as in the harmonic case the arbitrary constants P and J_2 (or J_1) can be fixed by imposing that the function $f(r)$ be C^1 on the surface of the star, i.e.,

$$[f]_\Sigma = [f']_\Sigma = 0 : \begin{cases} \left(r_0 - \dfrac{3}{2}m\right) J_1 + \bar{g}(r_0)\, J_2 = r_0 \bar{h}(r_0)\, \dfrac{P}{L} \\ \\ J_1 + \bar{g}'(r_0)\, J_2 = \left[\bar{h}(r_0) + r_0 \bar{h}'(r_0)\right] \dfrac{P}{L} \end{cases} \tag{79}$$

being

$$\bar{h}(r) \equiv F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{r^2}{L^2}\right) \tag{80}$$

These conditions lead to the following expressions for the constants

$$\begin{cases} J_2 = \dfrac{(r_0 - 3m/2)\left[\bar{h}(r_0) + r_0 \bar{h}'(r_0)\right] - r_0 \bar{h}(r_0)}{r_0 \bar{h}(r_0)[\bar{g}'(r_0) - 1] - \left[\bar{h}(r_0) + r_0 \bar{h}'(r_0)\right][\bar{g}(r_0) - (r_0 - 3m/2)]} \\ \\ \dfrac{P}{L} = \dfrac{(r_0 - 3m/2)\bar{g}'(r_0) - \bar{g}(r_0)}{r_0 \bar{h}(r_0)[\bar{g}'(r_0) - 1] - \left[\bar{h}(r_0) + r_0 \bar{h}'(r_0)\right][\bar{g}(r_0) - (r_0 - 3m/2)]} \end{cases} \tag{81}$$

whose expansion in power series of the parameter $\mu_0 \equiv 2m/r_0 = r_0^2/L^2$ is given by the following expressions

$$\begin{cases} 4m^2 J_2 = r_0^2 \left[\dfrac{-8}{5} + \dfrac{72}{35}\mu_0 - \dfrac{29}{1050}\mu_0^2 - \dfrac{8}{825}\mu_0^3 + O(\mu_0^4)\right] \\ \\ \dfrac{P}{L} = 1 - \mu_0 + \dfrac{9}{80}\mu_0^2 + \dfrac{1}{280}\mu_0^3 + O(\mu_0^4) \end{cases} \tag{82}$$

Finally, it should be mentioned that Teyssandier [10] have proved, in a similar way used here with the harmonic coordinates, that these values of the constants provide a C^1 metric on the surface Σ, i.e. , also the q-harmonic coordinates obtained are admissible coordinates in the sense of Lichnerowicz.

Appendix :

Heun's equation

A) The so called Heun's equation is the following second order linear differential equation

$$\begin{aligned} &x(x-1)(x-a)F''(x) + \{(\alpha+\beta+1)x^2 \\ &- [\alpha+\beta+1+a(\gamma+\delta)-\delta]x + a\gamma\}F'(x) + (\alpha\beta x - q)F(x) = 0 \end{aligned} \tag{A1}$$

where $(a,q;\alpha,\beta,\gamma,\delta)$ are numerical constants. This equation posses the following singular points which are all of them regular

$$\{x=0\,,x=1\,,x=a\,,x=\infty\} \tag{A2}$$

We restrict ourself to the point $x=0$, which is the one of interest in the text, and it is easyly obtained that the Frobenius series type solutions have exponents $(0\,,1-\gamma)$.

For the exponent zero the solution is

$$F(x)=H(a,q;\alpha,\beta,\gamma,\delta;x) \quad , \quad (\gamma\neq 0,-1,-2,\ldots) \tag{A3}$$

being H the Heun's series (31) appearing in the text, which is convergent if $|a|\geq 1$ and with a radius of convergence $|x|<1$.

For the exponent $1-\gamma$ the solution turns out to be

$$F(x)=x^{1-\gamma}H(a,q_1;\alpha_1,\beta_1,\gamma_1,\delta;x) \quad , \quad (\gamma\neq 1,2,3,\ldots) \tag{A4}$$

with

$$\begin{cases} q_1\equiv q+(1-\gamma)(\alpha+\beta+1-\gamma-\delta+a\delta) \\ \alpha_1\equiv\alpha+1-\gamma \quad , \quad \beta_1\equiv\beta+1-\gamma \quad , \quad \gamma_1\equiv 2-\gamma \end{cases} \tag{A5}$$

Now, if the previous series are not convergent, i.e., $|a|<1$, then it is performed the change of coordinates

$$x\longrightarrow\bar{x}=\bar{a}x \quad , \quad \left(\bar{a}\equiv\frac{1}{a}\right) \tag{A6}$$

and that leads to the following solution

$$\begin{cases} E_f=0: \quad F(x)=H(\bar{a},\bar{q};\alpha,\beta,\gamma,\bar{\delta};\bar{x}) \\ E_f=1-\gamma: \quad F(x)=x^{1-\gamma}H(\bar{a},\bar{q}_1;\alpha_1,\beta_1,\gamma_1,\bar{\delta};\bar{x}) \end{cases} \tag{A7}$$

being

$$\bar{q}\equiv\bar{a}\,q \quad , \quad \bar{\delta}\equiv\alpha+\beta+1-\gamma-\delta \quad , \quad \bar{q}_1\equiv\bar{a}\,q_1 \tag{A8}$$

B) The differential equation (26) appearing in the text can be reduced to the Heun's canonical form by means of the following change of function

$$f\longrightarrow F: \quad f(x)=x^k(x-2)^l(x-b)^m\,F(x) \tag{A9}$$

A rather unpleasent but straightforward calculation shows that there are four sets of suitable values for the exponents $(k\,,l\,,m)$ and consequently for the correspondent parameters $(a\,,q\,;\alpha\,,\beta\,,\gamma\,,\delta)$. Since we are looking for an analytical solution of the equation (A1) in a neighbourghood of the origin ($x = 0$), we put our attention into the eight possibilities provided by the two Frobenius' exponents $(0\,,1-\gamma)$ of the Heun's equation at this point. Taking into account that the unique right Frobenius' exponent from the beginning equation (26) is 1/2, finaly we have only four possibilities. It can be checked that, naturaly, all this four possibilities lead to the same analytical solution at $x = 0$, being one of them defined by the following set of parameters

$$E_f = 0 \quad \begin{cases} k = \dfrac{1}{2} \quad , \quad l = \dfrac{1}{2} \quad , \quad m = 0 \\ a = \dfrac{b}{2}\ ,\ q = 3\dfrac{b}{2} + \dfrac{1}{2}\ ;\ \alpha = 4\ ,\ \beta = 1\ ,\ \gamma = \dfrac{5}{2}\ ,\ \delta = \dfrac{5}{2} \end{cases} \tag{A10}$$

Now, since $|a| < 1$, we must resort to formulae (A6), (A8) to obtain the suitable values of the parameters

$$\bar{a} = \frac{2}{b} \quad , \quad \bar{q} = 3 + \frac{1}{b} \quad , \quad \alpha = 4 \quad , \quad \beta = 1 \quad , \quad \gamma = \frac{5}{2} \quad , \quad \bar{\delta} = 1 \tag{A11}$$

which construct the solution exhibited in the text (36).

References

[1] *See for instance*, A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics*, Benjamin Inc. (1967)

C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman & Co. San Francisco. (1970).

[2] V. A. Fock, *The Theory of Space, Time and Gravitation*, 2^o ed. revision Mc Millan. New York, translated by N. Kemmer. (1964)

Ll. Bel, J. Martin and A. Molina, *Journal of the Physical Society of Japan*, **63**, 4350–4363 (1994).

Recomendations I, II, III by the XXI General Assembly of the International Astronomic Union.

[3] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnetisme*, Masson & Cie. (1955)

[4] Quan-Hui Liu, *Journal of Math. Phys.*, **39**, 6086–6090. (1998).

L. A. Gegerly, *Journal of Math. Phys.*, **40**, 4177–4178. (1999).

Quan-Hui Liu, *Journal of Math. Phys.*, **40**, 4179. (1999).

[5] K. Heun, *Math Annalen* **33**, 161–179. (1889).

E. Kamke, *Differentialgleichungen. Lösungsmethoden und lösungen*, Chelsea Publishing Co. New York. 485–486. (1971).

E. L. Ince, *Ordinary differential equations*, Dover Publications Inc.. New York. 394. (1956).

[6] *See for instance*, C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman & Co. San Francisco. (1970).

[7] Preprint gr-qc/ 0104019. To appear in *Gen. Rel. and Grav.*

[8] Ll. Bel and J. Llosa, *Gen. Rel. and Grav.*, **27**, 1089–1110. (1995).

[9] J. M. Aguirregabiría, *private communication.* See also [7].

[10] P. Teyssandier, Preprint from Laboratoire de Gravitation et Cosmologie Relativistes. CNRS/ESA 7065 (France)

[11] *See for instance*, L. Landau et E. Lifchitz, *Théorie des Champs*, Editions MIR (Moscou 1970).

[12] K. S. Thorne, *Rev. Mod. Phys.*, **52**, 299–339 (1980)

Ll. Bel, T. Damour, N. Deruelle, J. Ibañez and J. Martín, *Gen. Rel. Grav.*, **13**, 963–1004 (1981).

L. Blanchet and T. Damour, *Phil. Trans. R. Soc. Lond.* **A 320**, 379–430 (1986).

[13] K. S. Thorne, *Rev. Mod. Phys.* **52**, 299. (1980).

[14] R. Geroch, *J. Math. Phys.*, **11**, 1955. (1970).

R. Geroch, *J. Math. Phys.*, **11**, 2580. (1970).

R. Geroch, *J. Math. Phys.*, **12**, 918. (1970)

R. O. Hansen, *J. Math. Phys.*, **15**, 46. (1974).